COMMUTATIONAL PROPERTIES OF POWERS OF OPERATORS OF MIXED TYPE INCREASING THE POWERS

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Abstract: The operators $Hf(z)=H_{p,q}f(z)=\dfrac{d^p}{dz^p}\left(z^q\int\limits_0^z f(\zeta)d\zeta\right)$ were considered in [11] with non-negative integers $p,q\in\mathbb{Z}_+$. Here they are considered with parameters $p<q+1$ in the space S of the polynomials with complex coefficients or in the space A_0 of functions analytic in neighbourhoods of the origin. Power series description of the commutant of powers H^s of H is given and then the question about the minimal commutativity of H^s (in the sense of Raichinov [13]) is discussed.

Key words: commutant of linear operator, minimal commutativity

JEL: C65, C69

КОМУТАЦИОННИ СВОЙСТВА НА СТЕПЕНИ НА ОПЕРАТОРИ ОТ СМЕСЕН ТИП ПОВИШАВАЩИ СТЕПЕНИТЕ

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Резюме: Операторите $Hf(z)=H_{p,q}f(z)=\dfrac{d^p}{dz^p}\left(z^q\int\limits_0^z f(\zeta)d\zeta\right)$ бяха разгледани в [11] при неотрицателни целочислени параметри $p,q\in\mathbb{Z}_+$. Тук те са разгледани за случая $p<q+1$ в пространството S на полиномите с комплексни коефициенти или в пространството A_0 на функциите, аналитични в околности на координатното начало. Дадено е описание на комутанта на степени H^s на H чрез степенни редове и е дискутиран въпросът за минималната комутативност на H^s (в смисъл на Райчинов[13]).

Ключови думи: комутант на линеен оператор, минимална комутативност.

JEL: C65, C69

1. Introduction

The commutational properties of linear operators are important not only in the functional analysis, but also in other fields of mathematics and also in different fields of physics, in particular the quantum physics, when the commuting of two operators could mean the possibility of simultaneous measuring of some quantities related to them. Commutation could be used also in other sciences.

Let A_0 be the space of functions analytic in neighbourhoods of the origin or its subspace S of the polynomials of the complex variable $z \in \mathbb{C}$. We want to consider a generalization of the usual operator of integration $\int_0^z f(\zeta)d\zeta$ multiplying it by a non-negative power z^q and then differentiating p times, i.e. we consider the operator of mixed type

$$Hf(z) = H_{p,q}f(z) = \frac{d^p}{dz^p}\left(z^q \int_0^z f(\zeta)d\zeta\right), \quad p,q \in \mathbb{Z}_+. \tag{1}$$

In [11] the commutational properties of the operator H in the case $p < q+1$ were investigated, and here similar results will be proved for arbitrarily fixed power H^s, $s \geq 1$, of the operator H.

It is suitable to represent first the action of the initial operator H on a single power z^k:

$$Hz^k = \frac{1}{k+1}(k+q+1)((k+q+1)-1)\ldots((k+q+1)-p+1)z^{(k+q+1)-p}.$$

Denoting $\alpha = q - p + 1$, we have $\alpha > 0$ and can write

$$Hz^k = a_k z^{k+\alpha}; \quad a_k = \frac{1}{k+1}\cdot\frac{(k+q+1)!}{(k+\alpha)!} \neq 0, \quad \alpha = q-p+1 > 0. \tag{2}$$

Now an arbitrary power H^s of H acts on z^k as

$$H^s z^k = a_k a_{k+\alpha}\ldots a_{k+(s-1)\alpha} z^{k+s\alpha}, \quad a_l = \frac{1}{l+1}\cdot\frac{(l+q+1)!}{(l+\alpha)!} \neq 0. \tag{3}$$

In order to avoid writing the long products in (3) we will use a short representation denoting them by one letter:

$$\beta_k = a_k a_{k+\alpha}\ldots a_{k+(s-1)\alpha}, \quad a_l = \frac{1}{l+1}\cdot\frac{(l+q+1)!}{(l+\alpha)!} \neq 0. \tag{4}$$

and then we can write simply

$$H^s z^k = \beta_k z^{k+s\alpha}, \quad \beta_k = \prod_{t=0}^{s-1} a_{k+(t-1)\alpha}, \quad a_l = \frac{1}{l+1}\cdot\frac{(l+q+1)!}{(l+\alpha)!} \neq 0. \tag{5}$$

In fact, if $y(z) = \sum_{k=0}^{\infty} b_k z^k$ is an analytic function from A_0 with coefficients $b_k = \frac{y^{(k)}(0)}{k!}$, then we have the short representation

$$H^s y(z) = \sum_{k=0}^{\infty} b_k \beta_k z^{k+\alpha} \tag{6}$$

with β_k from (4) and (5).

Let us give some definitions:

Definition 1. It is said that a continuous linear operator L commutes with a fixed operator M, if $LM = ML$. The set of all such operators is called the *commutant* of M and will be denoted by C_M.

Definition 2. It is said that a continuous linear operator T is generated by an operator M, if T is a polynomial of M with complex coefficients, i.e. $T = \sum_{n=0}^{\infty} d_n M^n$, $d_n \in \mathbb{C}$. The set of all operators generated by M will be denoted by G_M.

Obviously every operator T, which is generated by M, i.e. $T \in G_M$, also commutes with M, i.e. $T \in C_M$, and hence $G_M \subset C_M$. The opposite inclusion $G_M \supset C_M$ is, in general, not true. Therefore the following definition is natural:

Definition 3. (Raichinov [13]) An operator M is called minimally commutative if $G_M \supset C_M$, i.e. if the commutant C_M consists only of operators T generated by M and hence if $C_M = G_M$.

In our case the operator M will be the s-th power H^s of H defined by (5) and (6). In this paper we describe first the commutant C_{H^s} in the space A_0 of the functions analytic in (possibly different) neighbourhoods of the origin $z = 0$ in the complex plane $\mathbb{C}$. Next the question about the minimal commutativity of H^s in the sense of Rajchinov [13] will be considered.

Let us note that descriptions of commutants are made by many mathematicians. In the references of this paper we have included only a very small part of the publications related to the commutants of operators similar to the one considered here, see e.g. [1, 2, 3, 13, 14, 15, 4, 6, 7, 8, 9, 10, 11, 5, 12, 16, etc]. Additional huge number of publications related to commutants can be found in the bibliographies of the cited monographs.

2. Description of the commutant

Here we describe the commutant C_{H^s} of an arbitrarily fixed power H^s of the operator $H = H_{p,q}$, defined by (1) or (2):

Theorem 1. Let $H = H_{p,q}$ be the operator defined by (1) and (2), i.e.

$$Hy(z) = H_{p,q}y(z) = \frac{d^p}{dz^p}\left(z^q \int_0^z f(\zeta)d\zeta\right), \quad p,q \in \mathbb{Z}_+, \alpha = q - p + 1 > 0$$

$$Hy(z) = \sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} a_k z^{k+\alpha} \qquad \text{with} \qquad a_k = \frac{1}{k+1} \cdot \frac{(k+q+1)!}{(k+\alpha)!} \neq 0.$$

Then a continuous linear operator $L : A_0 \to A_0$ commutes with an arbitrary power H^s of the operator H given by (5) and (6) if and only if it has the form

$$Ly(z) = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{\left[\frac{m}{s\alpha}\right]s\alpha + s\alpha - 1} b_k \,. \frac{\beta_{m-s\alpha} \cdots \beta_{m-\left[\frac{k}{s\alpha}\right]s\alpha}}{\beta_{k-s\alpha} \cdots \beta_{k-\left[\frac{k}{s\alpha}\right]s\alpha}} c_{k-\left[\frac{k}{s\alpha}\right]s\alpha,\, m-\left[\frac{k}{s\alpha}\right]s\alpha} \right) z^m. \tag{7}$$

where $b_k = \dfrac{y^{(k)}(0)}{k!}$ and the complex numbers $c_{k,m}$ can be arbitrarily chosen for indices $0 \le k \le s\alpha - 1$ and $m = 0,1,2,\ldots$, but such that the series in (7) converges in some neighbourhood of the origin $z = 0$ of the complex plane $\mathbb{C}$.

Proof: We will look for a power series description of the action of an arbitrary operator L of the commutant C_{H^s}. Therefore we suppose that the operator L acts on an arbitrarily fixed power z^k as a power series

$$Lz^k = \sum_{m=0}^{\infty} c_{k,m} z^m \tag{8}$$

with unknown complex coefficients $c_{k,m}$.

First we express separately the two sides $LH^s z^k$ and $H^s L z^k$ of the commutational property $LH^s = H^s L$ for an arbitrarily fixed power z^k using (5):

$$LH^s z^k = L\beta_k z^{k+s\alpha} = \beta_k \sum_{m=0}^{\infty} c_{k+s\alpha,m} z^m = \sum_{m=0}^{\infty} \beta_k c_{k+s\alpha,m} z^m \tag{9}$$

$$HLz^k = H\sum_{m=0}^{\infty} c_{k,m} z^m = \sum_{m=0}^{\infty} c_{k,m}\beta_m z^{m+s\alpha} = \sum_{m=s\alpha}^{\infty} c_{k,m-s\alpha}\beta_{m-s\alpha} z^m. \tag{10}$$

We modified the index m in the last sum in (10).

Now we will use the identity theorem about the series expansion of analytic functions and will equate the coefficients of the powers z^m in (9) and (10).

The first observation is that in the case $0 \le m \le s\alpha - 1$, one has $\beta_k c_{k+s\alpha,m} = 0$. Since $\beta_k \ne 0$, if $k + s\alpha$ is replaced by k, it follows that

$$c_{k,m} = 0 \qquad \text{for } k \ge s\alpha,\ 0 \le m \le s\alpha - 1. \tag{11}$$

Next, in the other case $m \ge s\alpha$ it follows that $\beta_k c_{k+s\alpha,m} = c_{k,m-s\alpha}\beta_{m-s\alpha}$ and replacing $k + s\alpha$ by k, we obtain the following important recurrent relation:

$$c_{k,m} = \frac{\beta_{m-s\alpha}}{\beta_{k-s\alpha}} c_{k-s\alpha,m-s\alpha} \qquad \text{for } k \ge s\alpha,\ m \ge s\alpha.$$

Now we can use this recurrent relation (12) to express an arbitrary coefficient $c_{k,m}$ with indices $k \ge s\alpha$ and $m \ge s\alpha$ by some coefficient $c_{\text{щ},\mu}$, where either $0 \le \text{щ} \le s\alpha - 1$ or $0 \le \mu \le s\alpha - 1$.

Let us use the standard notation $[A]$ for the integer part of a number A. In particular, $\left[\dfrac{k}{s\alpha}\right]$ is the quotient and $k - \left[\dfrac{k}{s\alpha}\right]s\alpha$ is the remainder when k is divided by $s\alpha$.

In the case $\left[\dfrac{m}{s\alpha}\right] < \left[\dfrac{k}{s\alpha}\right]$, the recurrent formula (12) can be applied $\left[\dfrac{m}{s\alpha}\right]$ times and then

$$c_{k,m} = \frac{\beta_{m-s\alpha}}{\beta_{k-s\alpha}} c_{k-s\alpha,m-s\alpha} = \frac{\beta_{m-s\alpha}\beta_{m-2s\alpha}}{\beta_{k-s\alpha}\beta_{k-2s\alpha}} c_{k-2s\alpha,m-2s\alpha} =$$

$$= \ldots = \frac{\beta_{m-s\alpha}\cdots\beta_{m-\left[\frac{m}{s\alpha}\right]s\alpha}}{\beta_{k-s\alpha}\cdots\beta_{k-\left[\frac{m}{s\alpha}\right]s\alpha}} c_{k-\left[\frac{m}{s\alpha}\right]s\alpha,\, m-\left[\frac{m}{s\alpha}\right]s\alpha}. \tag{13}$$

In this case $0 \le m - \left[\dfrac{m}{s\alpha}\right] s\alpha \le s\alpha - 1$, i.e. the second index is the remainder when m is divided by $s\alpha$. Then by (11) the coefficient $c_{k-\left[\frac{m}{s\alpha}\right]s\alpha,\, m-\left[\frac{m}{s\alpha}\right]s\alpha}$ must be zero. Therefore, using (13), one has

$$c_{k,m} = 0, \quad \text{for } \left[\frac{m}{s\alpha}\right] < \left[\frac{k}{s\alpha}\right]. \tag{14}$$

In the other case, when $\left[\dfrac{m}{s\alpha}\right] \ge \left[\dfrac{k}{s\alpha}\right]$, one can apply $\left[\dfrac{k}{s\alpha}\right]$ times the recurrent formula (12) to get

$$\begin{aligned} c_{k,m} &= \frac{\beta_{m-s\alpha}}{\beta_{k-s\alpha}} c_{k-s\alpha,m-s\alpha} = \frac{\beta_{m-s\alpha}\beta_{m-2s\alpha}}{\beta_{k-s\alpha}\beta_{k-2s\alpha}} c_{k-2s\alpha,m-2s\alpha} = \\ &= \ldots = \frac{\beta_{m-s\alpha}\cdots\beta_{m-\left[\frac{k}{s\alpha}\right]s\alpha}}{\beta_{k-s\alpha}\cdots\beta_{k-\left[\frac{k}{s\alpha}\right]s\alpha}} c_{k-\left[\frac{k}{s\alpha}\right]s\alpha,\, m-\left[\frac{k}{s\alpha}\right]s\alpha}. \end{aligned} \tag{15}$$

This time the first index $k - \left[\dfrac{k}{s\alpha}\right] s\alpha$ is the remainder when k is divided by $s\alpha$.

Let us combine (14) and (15) as

$$c_{k,m} = \begin{cases} 0 & \text{for } \left[\dfrac{m}{s\alpha}\right] < \left[\dfrac{k}{s\alpha}\right], \\ \dfrac{\beta_{m-s\alpha}\cdots\beta_{m-\left[\frac{k}{s\alpha}\right]s\alpha}}{\beta_{k-s\alpha}\cdots\beta_{k-\left[\frac{k}{s\alpha}\right]s\alpha}} c_{k-\left[\frac{k}{s\alpha}\right]s\alpha,\, m-\left[\frac{k}{s\alpha}\right]s\alpha} & \text{for } \left[\dfrac{m}{s\alpha}\right] \ge \left[\dfrac{k}{s\alpha}\right]. \end{cases} \tag{16}$$

Now let us comment this important formula:

1. *All coefficients* $c_{k,m}$ *with* $0 \le k \le s\alpha - 1$ *can be chosen arbitrarily*,

and then

2. *All other coefficients* $c_{k,m}$ *with* $k \ge s\alpha$ *are either equal to zero or can be expressed by some of the arbitrarily chosen* $c_{\text{щ},\mu}$ *with* $0 \le \text{щ} = k - \left[\dfrac{k}{s\alpha}\right] s\alpha \le s\alpha - 1$*,* $\mu = m - \left[\dfrac{k}{s\alpha}\right] s\alpha \ge 0$.

The recurrent relation (16) allows to give the following representation of the action of the operator L from the commutant C_{H^s} on a single power z^k:

$$Lz^k = \sum_{m=\left[\frac{k}{s\alpha}\right]s\alpha}^{\infty} \frac{\beta_{m-s\alpha}\cdots\beta_{m-\left[\frac{k}{s\alpha}\right]s\alpha}}{\beta_{k-s\alpha}\cdots\beta_{k-\left[\frac{k}{s\alpha}\right]s\alpha}} c_{k-\left[\frac{k}{s\alpha}\right]s\alpha,\, m-\left[\frac{k}{s\alpha}\right]s\alpha} \,.\, z^m. \tag{17}$$

Finally, the action of an arbitrary operator $L \in C_H$ on any analytic function $y(z) = \sum_{k=0}^{\infty} b_k z^k \in A_0$ is

$$Ly(z) = L\sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} b_k L z^k =$$

$$= \sum_{k=0}^{\infty} b_k \left(\sum_{m=\left[\frac{k}{s\alpha}\right]s\alpha}^{\infty} \frac{\beta_{m-s\alpha}\cdots\beta_{m-\left[\frac{k}{s\alpha}\right]s\alpha}}{\beta_{k-s\alpha}\cdots\beta_{k-\left[\frac{k}{s\alpha}\right]s\alpha}} c_{k-\left[\frac{k}{s\alpha}\right]s\alpha,\, m-\left[\frac{k}{s\alpha}\right]s\alpha} \,.\, z^m \right). \tag{18}$$

Interchanging the summations, this can be written also as

$$Ly(z) = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{\left[\frac{m}{s\alpha}\right]s\alpha + s\alpha - 1} b_k \,.\, \frac{\beta_{m-s\alpha}\cdots\beta_{m-\left[\frac{k}{s\alpha}\right]s\alpha}}{\beta_{k-s\alpha}\cdots\beta_{k-\left[\frac{k}{s\alpha}\right]s\alpha}} c_{k-\left[\frac{k}{s\alpha}\right]s\alpha,\, m-\left[\frac{k}{s\alpha}\right]s\alpha} \right) z^m .$$

This is in fact the desired representation (7) of the commutant C_H^s of the s-th power H^s of the operator H.

Thus, we proved the *necessity*, i.e. if $L \in C_H^s$, then the operator L must be of the form (7).

Now, let us check the *sufficiency*, i.e. if an operator L has the form (7), then it commutes with the operator H^s, i.e. $LH^s = H^s L$.

It is enough to verify this for all powers z^k, $k = 0,1,2,\ldots$, since they form a basis in the space A_0 of the functions analytic in neighbourhoods of the origin $z = 0$ of the complex plane $\mathbb{C}$. In fact, for arbitrarily fixed k we can use the representation (17) instead of the general expression (7) and it is needed to check that $LH^s z^k = H^s L z^k$. First, the representation of $H^s L z^k$ is

$$H^s L z^k = \sum_{m=\left[\frac{k}{s\alpha}\right]s\alpha}^{\infty} \frac{\beta_{m-s\alpha}\cdots\beta_{m-\left[\frac{k}{s\alpha}\right]s\alpha}}{\beta_{k-s\alpha}\cdots\beta_{k-\left[\frac{k}{s\alpha}\right]s\alpha}} c_{k-\left[\frac{k}{s\alpha}\right]s\alpha,\, m-\left[\frac{k}{s\alpha}\right]s\alpha} \,.\, \beta_m z^{m+s\alpha} .$$

A replacement of m by $m - s\alpha$ is suitable for future comparison:

$$H^s L z^k = \sum_{m=\left[\frac{k+s\alpha}{s\alpha}\right]s\alpha}^{\infty} \frac{\beta_{m-s\alpha}\cdots\beta_{m-\left[\frac{k+s\alpha}{s\alpha}\right]s\alpha}}{\beta_{k-s\alpha}\cdots\beta_{k-\left[\frac{k}{s\alpha}\right]s\alpha}} c_{k-\left[\frac{k}{s\alpha}\right]s\alpha,\, m-\left[\frac{k+s\alpha}{s\alpha}\right]s\alpha} \,.\, z^m . \tag{20}$$

Next, the representation of $LH^s z^k$ is

$$LH^s z^k = L(\beta_k z^{k+s\alpha}) = \beta_k L z^{k+s\alpha} =$$

$$= \beta_k \,.\, \sum_{m=\left[\frac{k+s\alpha}{s\alpha}\right]s\alpha}^{\infty} \frac{\beta_{m-s\alpha}\cdots\beta_{m-\left[\frac{k+s\alpha}{s\alpha}\right]s\alpha}}{\beta_{k+s\alpha-s\alpha}\cdots\beta_{k+s\alpha-\left[\frac{k+s\alpha}{s\alpha}\right]s\alpha}} c_{k+s\alpha-\left[\frac{k+s\alpha}{s\alpha}\right]s\alpha,\, m-\left[\frac{k+s\alpha}{s\alpha}\right]s\alpha} \,.\, z^m \tag{21}$$

and after canceling the first factor in the denominator with β_k in front of the sum it is obvious that (20) and (21) are one and the same. Thus $H^s L z^k = LH^s z^k$ for arbitrary power z^k and the *sufficiency* of (7) is also proved.

2. Minimal commutativity

In order to investigate if an arbitrary power H^s of the operator H, defined by (1), about minimal commutativity in the sense of Rajchinov [13] (our Definition 3), it is needed to describe the operators T generated by H^s:

Theorem 2. If $T=\sum_{n=0}^{\infty} d_n (H^s)^n = \sum_{n=0}^{\infty} d_n H^{ns} \in G_{H^s}$ is an operator, generated by the operator H^s (defined by (5) or (6)), then its action on the powers z^k has the form

$$Tz^k = d_0 z^k + \sum_{n=1}^{\infty}\left(d_n \prod_{t=0}^{n-1} \beta_{k+ts\alpha} \right) z^{k+ns\alpha}. \tag{22}$$

Proof: $H^{ns}z^k$ has the following expression:

$$H^{ns}z^k = \beta_k \beta_{k+s\alpha} \dots \beta_{k+(n-1)s\alpha} z^{k+ns\alpha}$$

Therefore

$$Tz^k = \sum_{n=0}^{\infty} d_n H^{ns} z^k = d_0 z^k + d_1 \beta_k z^{k+s\alpha} + d_2 \beta_k \beta_{k+s\alpha} z^{k+2s\alpha} +$$

$$+\dots + d_n \beta_k \beta_{k+s\alpha} \dots \beta_{k+(n-1)s\alpha} z^{k+ns\alpha} + \dots = d_0 z^k + \sum_{n=1}^{\infty}\left(d_n \prod_{t=0}^{n-1} \beta_{k+ts\alpha} \right) z^{k+ns\alpha}, \tag{23}$$

which is the desired representation (22).

One can immediately describe the action on any analytic function $y \in A_0$:

Theorem 3. If $T=\sum_{n=0}^{\infty} d_n (H^s)^n = \sum_{n=0}^{\infty} d_n H^{ns} \in G_H^s$ is an operator, generated by the operator H^s and $y \in A_0$, then

$$Ty(z) = \sum_{m=0}^{\infty}\left[\frac{y^{(m)}(0)}{m!} d_0 + \sum_{l=1}^{\left[\frac{m}{s\alpha}\right]} \left(\frac{y^{(m-ls\alpha)}(0)}{(m-ls\alpha)!} \prod_{t=1}^{l} \beta_{m+ts\alpha} \right) d_l \right] z^m. \tag{24}$$

Proof: From Theorem 2

$$Ty(z) = \sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} Tz^k = \sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} \left(d_0 z^k + \sum_{n=1}^{\infty}\left(d_n \prod_{t=0}^{n-1} \beta_{k+ts\alpha} \right) z^{k+ns\alpha} \right). \tag{25}$$

If $b_k = \dfrac{y^{(m)}(0)}{m!}$, $k = 0,1,2,\dots$, be a short notation of the coefficients of y, then we can represent (25) in a different way which is more convenient for future comparison. We can gather the equal powers of z at one place as follows:

$$Ty(z) = \sum_{m=0}^{\infty}\left[b_m d_0 + \sum_{l=1}^{\left[\frac{m}{s\alpha}\right]} \left(b_{m-ls\alpha} d_l \prod_{t=1}^{s} \beta_{m+ts\alpha} \right) \right] z^m.$$

This is in fact the desired representation (24).

The main theorem in this section is a necessary and sufficient condition for minimal commutativity of H^s:

Theorem 4. Let $H = H_{p,q}$ be the operator defined by (1) and H^s be an arbitrary power of H presented in (5) and (6). Then the operator H^s is minimally commutative if and only if $s\alpha = s(q-p+1) = 1$, i.e when $s=1$ and $p=q$.

Proof: Instead of considering the action of the operator H^s on an arbitrary function $y \in A_0$, given in Theorem 1, it is enough to fix arbitrarily an integer k and to compare the representation (22) of Tz^k from Theorem 3 with (17), which is an expression of Lz^k from the description of the commutant:

$$Lz^k = \sum_{m=\left[\frac{k}{s\alpha}\right]s\alpha}^{\infty} \frac{\beta_{m-s\alpha}\cdots\beta_{m-\left[\frac{k}{s\alpha}\right]s\alpha}}{\beta_{k-s\alpha}\cdots\beta_{k-\left[\frac{k}{s\alpha}\right]s\alpha}} c_{k-\left[\frac{k}{s\alpha}\right]s\alpha, m-\left[\frac{k}{s\alpha}\right]s\alpha} \,.\, z^m. \tag{27}$$

If $s\alpha \geq 2$, note that for $0 \leq k \leq s\alpha - 1$ and $0 \leq m \leq s\alpha - 1$ only the power z^k in the description of the generated operators (22)

$$Tz^k = d_0 z^k + \sum_{n=1}^{\infty}\left(d_n \prod_{t=0}^{n-1} \beta_{k+ts\alpha} \right) z^{k+ns\alpha} \tag{28}$$

could have a coefficient d_0 different from zero, while all $(s\alpha)^2 \geq 4$ coefficients $c_{k,m}$ in (27) could be chosen arbitrarily, in particular different from zero. This shows that

If $s\alpha \geq 2$, then the general operator H^s is not minimally commutative.

Hence H could be minimally commutative only in the case $s\alpha = 1$. In this case we want to check whether Lz^k has the form of Tz^k for arbitrary k. One has

$$Lz^k = \sum_{m=k}^{\infty} \frac{\beta_{m-1}\cdots\beta_{m-k}}{\beta_{k-1}\cdots\beta_0} c_{0,m-k} \,.\, z^m. \tag{29}$$

$$Tz^k = d_0 z^k + \sum_{n=1}^{\infty}\left(d_n \prod_{t=0}^{n-1} \beta_{k+t} \right) z^{k+n}. \tag{30}$$

Replacing $k+n$ by m, (30) becomes

$$Tz^k = d_0 z^k + \sum_{m=k+1}^{\infty}\left(d_{m-k} \prod_{t=0}^{m-k-1} \beta_{k+t} \right) z^{m}. \tag{31}$$

Now we can equate the coefficients of the equal powers $z^m, m = k, k+1, \ldots$ in (29) and (31) and after cancelling the common factors we obtain the following system:

$$\begin{aligned} c_{0,0} &= d_0 \\ c_{0,1} &= d_1\beta_0 \\ c_{0,2} &= d_2\beta_0\beta_1 \\ \cdots &= \cdots \\ c_{0,s} &= d_s\beta_0\beta_1\cdots\beta_{s-1} \\ \cdots &= \cdots \end{aligned}$$

In fact, these equations allow solving with respect to d_s, $s = 0,1,2,\ldots$, and we can write the following formulae relating the arbitrarily chosen coefficient $c_{0,s}$ in the description of the commutant C_{H^s} with the coefficients d_s in the description of the generated by H^s operators in G_{H^s}:

$$c_{0,s} = d_s\beta_0\beta_1\cdots\beta_{s-1}, \qquad d_s = c_{0,s}\frac{1}{\beta_0\beta_1\cdots\beta_{s-1}}, \qquad s = 0,1,2,\ldots$$

These formulae do not depend on the arbitrarily chosen k and thus:

In the case $s\alpha = 1$*, i.e when* $s = 1$ *and* $p = q$*, any operator* L *from the commutant* C_{H^s} *can be treated as an operator generated by* H^s*, i.e.* $L \in G_{H^s}$*,* $C_{H^s} \subset G_{H^s}$ *and the operator* H^s *is minimally commutative.*

4. Final remarks

We made the description of the commutant in the space A_0 of the functions $y(z) = \sum_{k=0}^{\infty} b_k z^k$ analytic in a neighbourhood of the origin. In this case the coefficients

$$\beta_k = a_k a_{k+\alpha} \ldots a_{k+(s-1)\alpha}, \quad a_l = \frac{1}{l+1} \cdot \frac{(l+q+1)!}{(l+\alpha)!} \neq 0$$

from (5) satisfy $\limsup_{k\to\infty} \sqrt[k]{|\beta_k|} < \infty$ since the radius of convergence of the series is positive

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|\beta_k|}} > 0$$

and then

$$\limsup_{k\to\infty} \sqrt[k]{|b_k \beta_k|} \leq \limsup_{k\to\infty} \sqrt[k]{|b_k|} . \limsup_{k\to\infty} \sqrt[k]{|\beta_k|} < \infty.$$

This observation shows that the series in all descriptions are convergent in the space A_0 under the restriction the arbitrarily chosen $c_{k,m}$, $0 \leq m \leq s\alpha - 1$, to be bounded by a constant, or more generally, to have such a growth with respect to k which ensures convergence in (8).

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